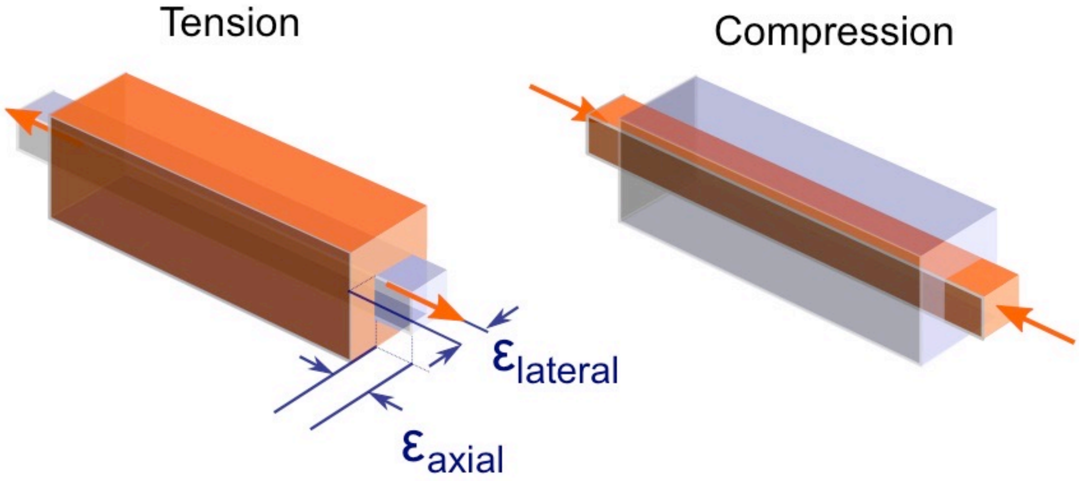
Strain and stress in higher dimensions

3D response to 1D strain

The Poisson effect

- In chapter 2 we've discussed how loads introduce stress and strain in the the direction that the load acts (1D problems)
- In reality, a strain in one direction (*axial strain*) also induces strains in the two perpendicular directions (*lateral strain*)



3D response to 1D strain
The Poisson effect

$$\boxed{\varepsilon_{lateral} = -\nu \cdot \varepsilon_{axial}}$$

$$\nu := \left| \frac{\varepsilon_{lateral}}{\varepsilon_{axial}} \right| = \text{Poisson ratio}$$

$$\boxed{G = \frac{E}{2(1+\nu)}}$$

The lateral strain can be described using the *Poisson equation.*

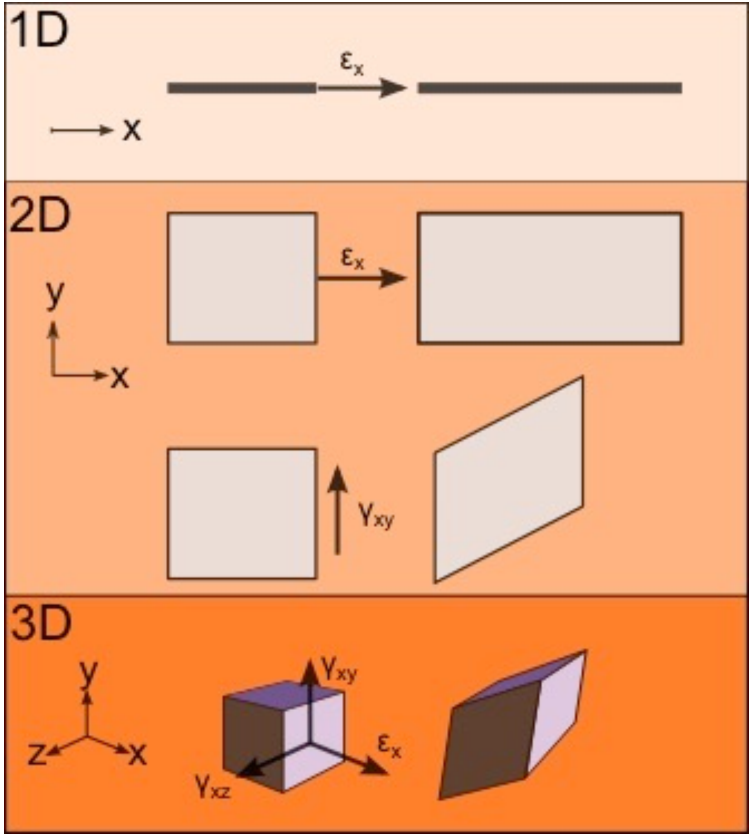
ν is a materials property and is on the order of 0.25-0.35 for most materials (0.1-0.5)

IMPORTANT: The Poisson effect does NOT cause any additional stress in the material, unless the transverse displacement is prevented

For Hookean materials, the Poisson ratio relates the *elastic modulus E* and the *shear modulus G.*

The Strain Tensor

"Where did you get yor chiropractic license?"



The strain tensor

In real world problems, we need to keep track of normal strains ε and shear strains γ in multiple directions

On each face of a (virtual) cube, we can have one normal strain and two shear strains

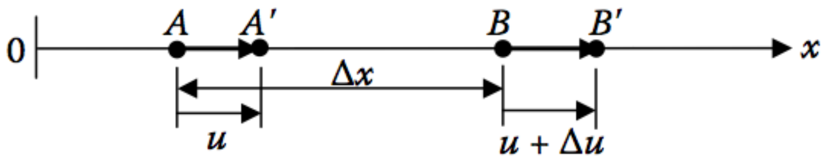
For 3 dimensions (3 faces) this results in 3 normal strains and 6 shear strains

The 9 strains can be conveniently combined into the strain tensor

REMEMBER: Strain is a local property and can change dramatically in the material. That's why we need a definition of the 3D strain based on an infinitesimal element.

Review: The strain tensor

Infinitesimal definition of strain



$$\boxed{\varepsilon = \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} = \frac{du}{dx}}$$

- Like in the 1D case, we derive the definition of strain from the stretching of a segment AB with the original length of Δx that is part of a larger line.
- *u* is the “rigid body motion” and *Δu* is the stretching of the element *Δx*

The strain tensor

Definition of *normal* strain in 3 dimensions

- The three normal components of the strain are then:

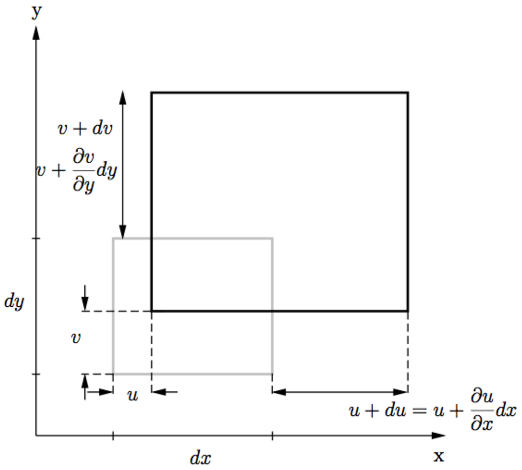
$$\begin{aligned} \varepsilon_x &= \varepsilon_{xx} = \frac{\partial u}{\partial x} \\ \varepsilon_y &= \varepsilon_{yy} = \frac{\partial v}{\partial y} \\ \varepsilon_z &= \varepsilon_{zz} = \frac{\partial w}{\partial z} \end{aligned}$$

- The 3 normal changes show the change in shape of the parallelepiped with initial volume *dxdydz.*
- The normalized volume change is the sum of the normalized strains

$$\frac{\Delta V}{V} = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

Deriving the strain tensor

Definition of normal strain in 3 dimensions



- Analog to the 1D case we can define the 3 normal components of the strain.

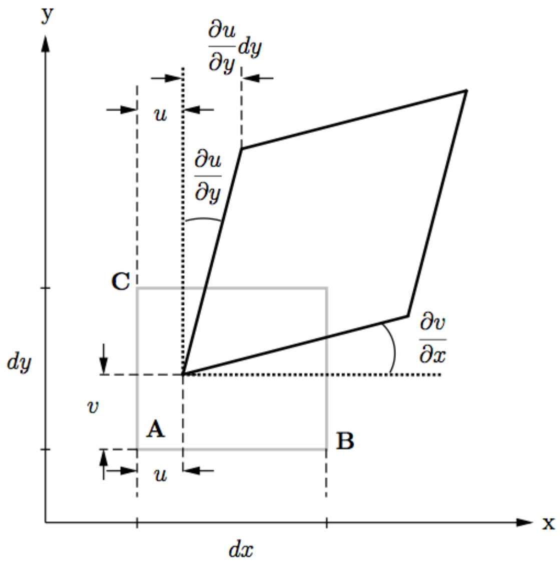
$$\varepsilon_x = \varepsilon_{xx} = \frac{\partial u}{\partial x}$$
$$\varepsilon_y = \varepsilon_{yy} = \frac{\partial v}{\partial y}$$
$$\varepsilon_z = \varepsilon_{zz} = \frac{\partial w}{\partial z}$$

- The 3 normal changes show the change in shape of the parallelepiped with initial volume *dxdydz.*
- The normalized volume change is the sum of the normalized strains

$$\frac{\Delta V}{V} = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

The strain tensor

Definition of shear strain in 3 dimensions



- The slopes of initially horizontal line is $\frac{\partial v}{\partial x}$
- The slopes of initially vertical line is $\frac{\partial u}{\partial y}$
- Therefore the change in the previously right angle, which is the shear strain is:

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

- From symmetry and extrapolation into the third dimension we learn that:

$$\boxed{\begin{aligned} \gamma_{xy} = \gamma_{yx} &= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \\ \gamma_{xz} = \gamma_{zx} &= \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \\ \gamma_{yz} = \gamma_{zy} &= \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \end{aligned}}$$

The strain tensor

Nomenclature

$$\overleftrightarrow{\varepsilon} = \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{pmatrix} = \begin{pmatrix} \varepsilon_{xx} & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{xy} & \varepsilon_{yy} & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{xz} & \frac{1}{2}\gamma_{yz} & \varepsilon_{zz} \end{pmatrix} =$$

- The shear components of the strain tensor are DEFINED to be ½ of the engineering strain.
- THIS IS AN ENDLESS SOURCE OF CONFUSION!

$$\overleftrightarrow{\varepsilon} = \begin{pmatrix} \varepsilon_{xx} & \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \\ \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & \varepsilon_{yy} & \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) \\ \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) & \varepsilon_{zz} \end{pmatrix}$$

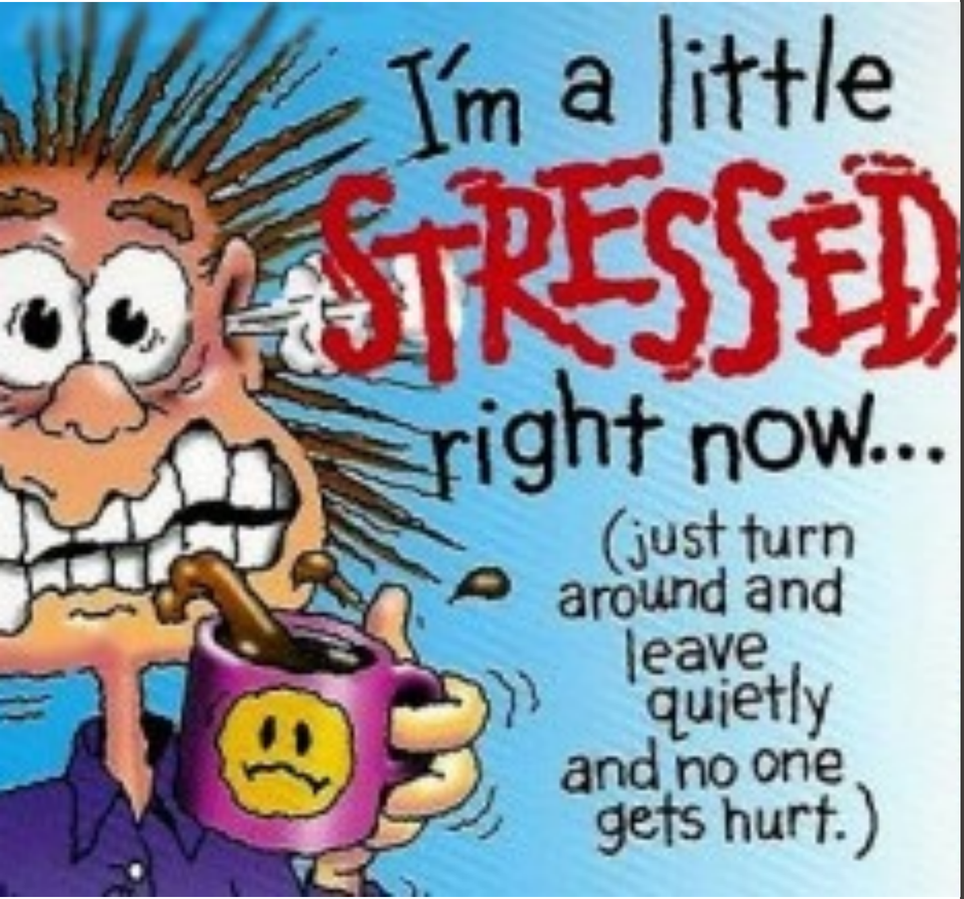
The strain tensor

Nomenclature

- We can write the 9 components of the strain as a second order tensor (for each strain we have a direction, a magnitude and relative to which plane our stress is quantified).
- Example: ε_{xx} represents the magnitude of deformation in the x direction, relative to a reference length in the x direction.
- To make the strain tensor behave mathematically like a proper tensor, the shear components of the strain tensor are defined as HALF the engineering strain.

$$\varepsilon_{kl} = \frac{1}{2}(u_{l,k} + u_{k,l}) = \frac{1}{2}\left(\frac{\partial u_l}{\partial x_k} + \frac{\partial u_k}{\partial x_l}\right)$$

- This equation defines 6 independent components of a symmetric, second order tensor ($x_i \in \{x,y,z\}$, $u_i \in \{u,v,w\}$)



The Stress Tensor

The stress tensor

Nomenclature

- We can write the individual components of the stress state also in tensor form
- The first sub script denotes the normal vector of the area in question, and the second subscript denotes the direction of the force component that acts on the area

$$\sigma_{yx} = \lim_{\Delta A_y \to 0} \frac{\Delta F_x}{\Delta A_y} \qquad \sigma_{yy} = \lim_{\Delta A_y \to 0} \frac{\Delta F_y}{\Delta A_y}$$

- The stress vector equation can then be written as:

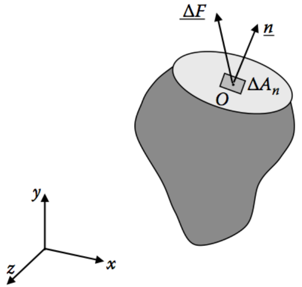
$$\vec{\sigma}_x = \sigma_{xx} \cdot \vec{e}_x + \sigma_{xy} \cdot \vec{e}_y + \sigma_{xz} \cdot \vec{e}_z$$
$$\vec{\sigma}_y = \sigma_{yx} \cdot \vec{e}_x + \sigma_{yy} \cdot \vec{e}_y + \sigma_{yz} \cdot \vec{e}_z$$
$$\vec{\sigma}_z = \sigma_{zx} \cdot \vec{e}_x + \sigma_{zy} \cdot \vec{e}_y + \sigma_{zz} \cdot \vec{e}_z$$

The stress tensor

Intensity of forces on internal surfaces

- *Stress is the intensity of internal forces (**F/A**) on (virtual) surfaces within a body subject to loads **P**. (**F**, **A**, and **P** are vectors)*
- *We can therefore define the stress vector **σ**$_n$ as:*

$$\vec{\sigma_n} = \lim_{\Delta A_n \to 0} \frac{\Delta \vec{F}}{\Delta A_n}$$



- *In Cartesian coordinates:*

$$\vec{\sigma_n} = \lim_{\Delta A_n \to 0} \frac{\Delta F_x \vec{e_x} + \Delta F_y \vec{e_y} + \Delta F_z \vec{e_z}}{\Delta A_n}$$

- *For surfaces perpendicular to the x,y or z axis:*

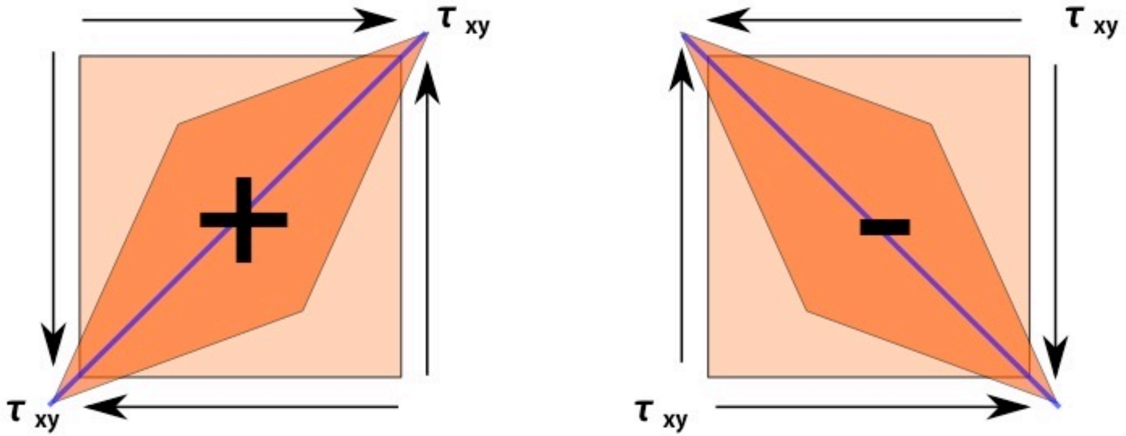
$$\vec{\sigma_x} = \lim_{\Delta A_x \to 0} \frac{\Delta F_x \vec{e_x} + \Delta F_y \vec{e_y} + \Delta F_z \vec{e_z}}{\Delta A_x}$$

$$\vec{\sigma_y} = \lim_{\Delta A_y \to 0} \frac{\Delta F_x \vec{e_x} + \Delta F_y \vec{e_y} + \Delta F_z \vec{e_z}}{\Delta A_y}$$

$$\vec{\sigma_z} = \lim_{\Delta A_z \to 0} \frac{\Delta F_x \vec{e_x} + \Delta F_y \vec{e_y} + \Delta F_z \vec{e_z}}{\Delta A_z}$$

The stress tensor – sign convention

- Normal stress: normal stress is considered positive if it puts an element in tension, and negative if it puts an element in compression
- Shear stress:



The stress tensor

Nomenclature

- The 9 stress components can be again combined to a tensor:

$$\overleftrightarrow{\tau} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} = \begin{pmatrix} \sigma_{x} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{y} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{z} \end{pmatrix}$$

- σ_i are normal stresses, τ_{ij} are shear stresses
- The stress tensor is symmetric: $\tau_{xy} = \tau_{yx}$
- Both stress and strain tensor are 2nd order tensors and can be diagonalized to give the *principal values* (extreme values of stress and strain)

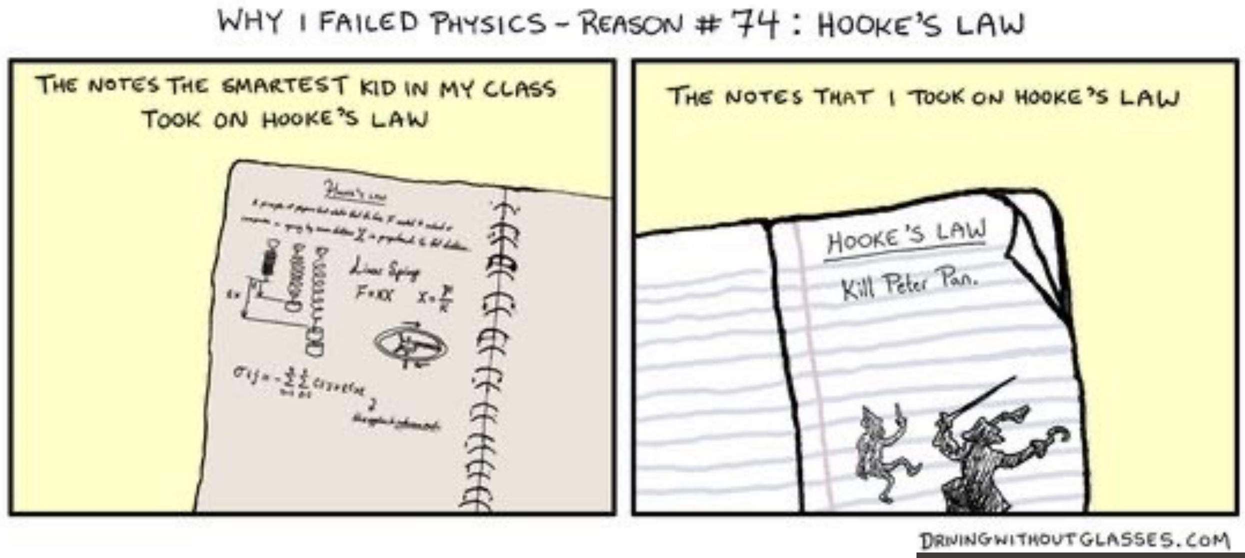
The stress tensor

Final remarks

If we pass three mutually orthogonal planes through any given point O and find the stress vectors on each of the three mutually perpendicular faces drawn through O, then we have fully characterized the stress at point O

Due to the symmetry of the stress tensor, we only need 6 stress values to fully describe the stress state

Stress represents the intensity of internal forces on surfaces within a body subject to loads. At an imaginary cut or section, a vector sum of these forces keeps the body in equilibrium



Hooke's Law in 3D

Hooke's Law in 3D

- In the 1D case, Hooke's law has the form: $\sigma = E \cdot \varepsilon$

$$\tau = G \cdot \gamma$$

- In 3D we do the same thing, but with ε,γ,σ,τ replaced by the second order tensors **ε** and **τ:**

$$\overleftrightarrow{\tau} = \overset{\Leftrightarrow}{C}\, \overleftrightarrow{\varepsilon}$$

$$\begin{pmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{pmatrix} = \overset{\Leftrightarrow}{C} \begin{pmatrix} \varepsilon_x & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{xy} & \varepsilon_y & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{xz} & \frac{1}{2}\gamma_{yz} & \varepsilon_z \end{pmatrix}$$

- Or in index notation: $\tau_{ij} = C_{ijkl} \cdot \varepsilon_{kl}$
- C_{ijkl} is called the *stiffness*, C_{ijkl}^{-1} is called the *compliance*
- A tensor of rank 4 in 3D space has 3x3x3x3=**81 elements**!

Generalized Hooke's law

Simplification of the stiffness matrix

Mathematically general form	81 independent components	
Take into account the symmetry of τ and ε	36 independent components	
symmetry due to the existence of strain energy U_0	21 independent components	General case for anisotropc elasticity
isotropic case: C_{ijkl} is invariant to coordinate rotation	2 independent components	Case for isotropic materials (materials that have same E and G in all directions)

Generalized Hooke's law

Full form for isotropic homogeneous materials

- Because of the symmetry we can simplify:

$$\begin{pmatrix} \varepsilon_x & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{xy} & \varepsilon_y & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{xz} & \frac{1}{2}\gamma_{yz} & \varepsilon_z \end{pmatrix} = \overleftrightarrow{C}^{-1} \begin{pmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{pmatrix}$$

- We rearrange the non redundant components:

$$\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{pmatrix} = \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ 2\varepsilon_{xy} \\ 2\varepsilon_{xz} \\ 2\varepsilon_{yz} \end{pmatrix} = \begin{pmatrix} \text{some 6x6 marix} \\ \\ \\ \\ \\ \end{pmatrix} \cdot \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{xz} \\ \tau yz \end{pmatrix}$$

- These are no longer the stress or strain tensors! They are just a simpler way to write Hooke's law in 3D.

Generalized Hooke's law

For isotropic cases

- For homogeneous isotropic materials we can write Hooke's law for the normal stresses and for the shear stresses as:

$$\varepsilon_x = \varepsilon_{xx} = \frac{1}{E}(\sigma_x - \nu \cdot \sigma_y - \nu \cdot \sigma_z)$$
$$\varepsilon_y = \varepsilon_{yy} = \frac{1}{E}(-\nu \cdot \sigma_x + \sigma_y - \nu \cdot \sigma_z)$$
$$\varepsilon_z = \varepsilon_{zz} = \frac{1}{E}(-\nu \cdot \sigma_x - \nu \cdot \sigma_y + \sigma_z)$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G}$$
$$\gamma_{xz} = \frac{\tau_{xz}}{G}$$
$$\gamma_{yz} = \frac{\tau_{yz}}{G}$$

- Since E and G are related through v, there are only 2 independent materials properties in these equations.

$$G = \frac{E}{2(1+\nu)}$$

Generalized Hooke's law

Full form for isotropic homogeneous materials

$$\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{pmatrix} = \frac{1}{E} \begin{pmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu) \end{pmatrix} \cdot \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{pmatrix}$$

Generalized Hooke's law

Volume changes

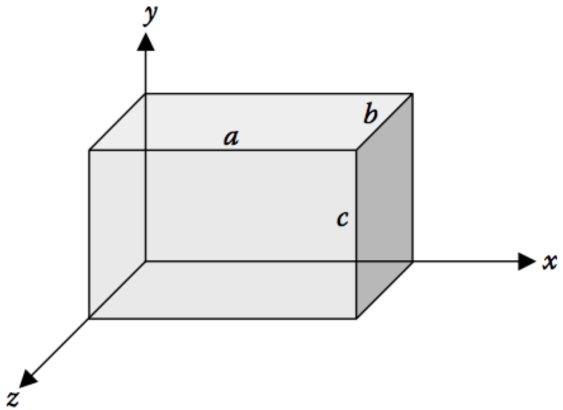
- From the definition of the infinitesimal strains, we can calculate the change in volume to be

$$
\begin{aligned}
\frac{\Delta V}{V} &= \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} \\
&= \frac{1-2\nu}{E}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}) \\
&= \frac{1}{K}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz})
\end{aligned}
$$

- *K* is the *bulk modulus* of the material. It can be calculated from the two independent materials variables E and v

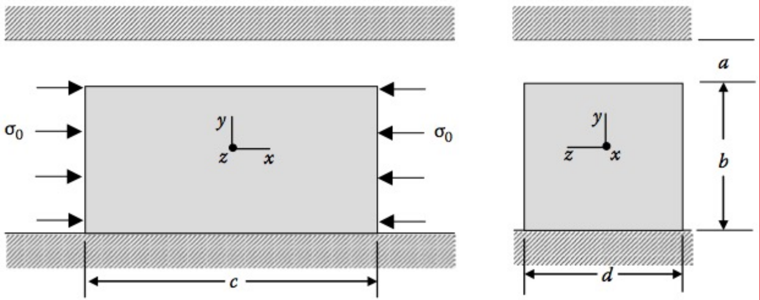
Example 3.1

Triaxial loading



A rectangular copper alloy block as shown in the figure has the following dimensions: a = 200 mm, b = 120 mm, and c = 100 mm. This block is subjected to a triaxial loading in equilibrium having the following magnitude: σx = +2.40 MPa, σy = –1.20 MPa, and σz = –2.0 MPa. Assuming that the applied forces are uniformly distributed on the respective faces, determine the size changes that take place along a, b, and c. Let E = 140 GPa and v = 0.35.

Example 3.2



A rectangular block is compressed by a uniform stress σ_0 as it sits between two rigid surfaces with the gap *a* shown in Figure 3.14. Determine (a) the stress σ_{yy}; (b) the change in the length along the *x* axis as the gap *a* is closed; and (c) the minimum value of σ_0 need to close the gap and the change in length of c when the gap is just closed. Assume $\sigma_{zz}=0$